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FX Instalment Options – A Closed-Form Solution and the Limiting Case

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1. Instalment Option

1.1. Definition

- Like Vanilla Option, but
  1. (1) Premium is divided into several payments and is paid periodically on so-called "instalment dates"
  2. (2) Holder has the right to cancel option through the termination of instalment payments

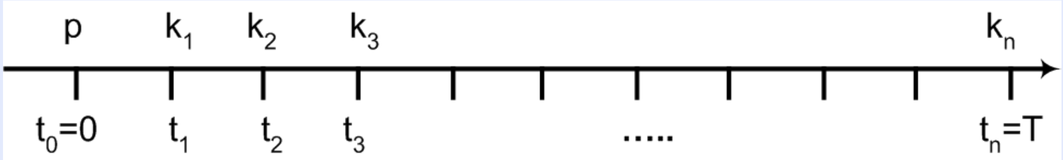

Figure 1: Dates for Instalment Payments

- Other names: continuation option, pay-as-you-go option, a generalization of compound option

- n-Instalment Option can be understood as a series of n options depending on each other

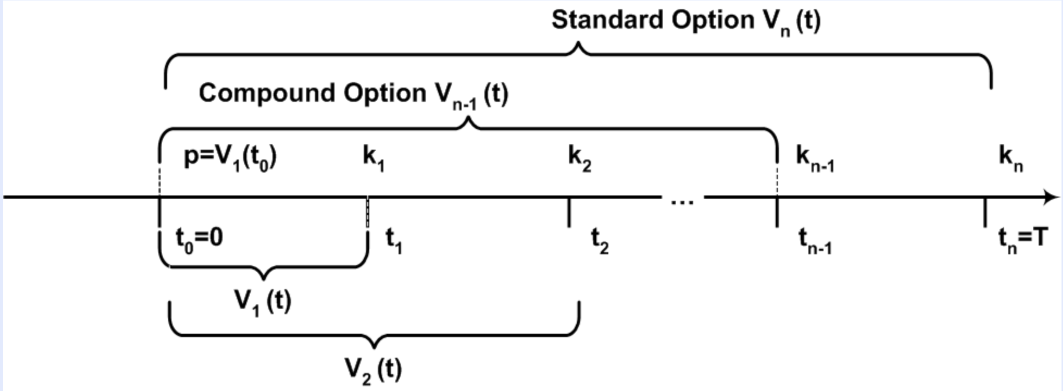

Figure 2: Lifetimes of the options V_i

- Characterized by
  - n exercise times $t_1, ..., t_n = T$ (often $t_i = iT/n$ for all i),
  - n strike prices $k_1, ..., k_n$,
  - n put/ call indicators $\phi_1, ..., \phi_n$ where $\phi_i \triangleq \begin{cases} +1 & \text{if option } i \text{ is a call} \\ -1 & \text{if option } i \text{ is a put} \end{cases}$

Market data

- S_0: spot
- r_d: domestic interest rate
- r_f: foreign interest rate
- σ: volatility

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1.2. Advantages of Instalment Options

- Traded over-the-counter tailor-made to client needs
- Prevention of losses through possibility of termination
- Helpful in situations where necessity of hedge is uncertain
- Low initial premium is easy to schedule in the firm's budget

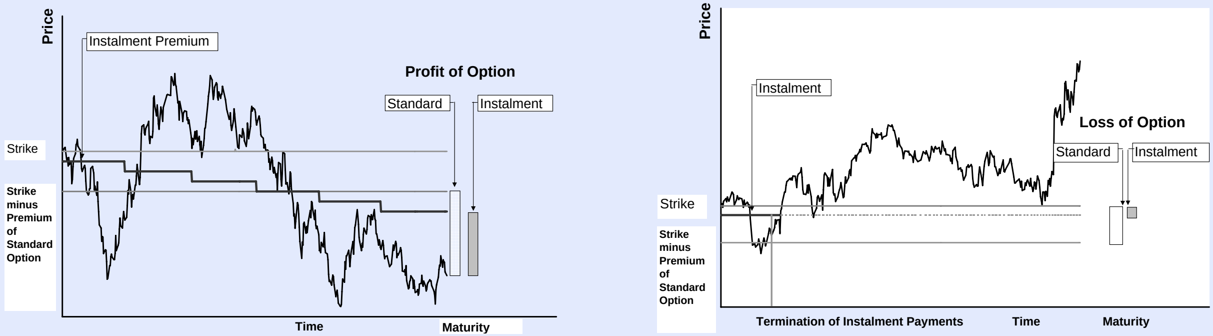

Figure 3: Comparison of Instalment Option with Vanilla Put: Continuation of instalment payments until expiration vs. Continuation of instalment payments until expiration, reference [1]

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1.3. Example of a Traded Instalment Option

- Application area: International Treasury Management
- Corporate buys EUR Call/ USD Put 25 Mio EUR notional
- Strike price: 1.0500 EUR/USD
- Exercise type: European
- Maturity date: 17 Dec 2003, Delivery settlement on 19 Dec 2003
- Transaction date: 19 Dec 2002
- EUR USD spot ref: 1.0259
- Premium and strike prices: 285,500.00 USD
- Decision and Value dates: 31/03/03, 02/04/03, 30/06/03, 02/07/03, 30/09/03, 02/10/03
- The corporate has extended the instalment at all dates and finally sold the EUR call on Nov 19 2003 for a profit of 2.77 MIO EUR (spot was at 1.1900).

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1.4. Literature on Instalment Option

Davis, Tompkins and Schachermayer (2001) derive no-arbitrage bounds on the price of Instalment options, which are used to set up static hedges and to compare them to dynamic hedging strategies [7].

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Fouque and Han (2005) examine the case of Compound options ($n = 2$) in stochastic volatility models [8].

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2. Pricing of Instalment Options in the Black-Scholes Model

- Like Vanilla Options or Compound Options, i.e. discounted expectation of payoff function
- $dS_t = S_t[(r_d - r_f)dt + \sigma dW_t]$ for $0 \leq t \leq T$
  $S_{t_2} = S_{t_1} \exp((r_d - r_f - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}Z)$, for $0 \leq t_1 \leq t_2 \leq T$,
  $\Delta t = t_2 - t_1$

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  $\Delta t = t_2 - t_1$
- Payoff at maturity is $\max(\phi_n(S_T - k_n), 0) \overset{def}{=} (\phi_n(S_T - k_n))^+$

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- Payoff at maturity is $\max(\phi_n(S_T - k_n), 0) \overset{def}{=} (\phi_n(S_T - k_n))^+$
- Date before last instalment date t_{n-1} buyer pays k_{n-1} to receive classical european option, in which the price at t_{n-1} is described by

$$V_n(s) \overset{def}{=} V_{Std}(s) = e^{-r_d(t_n - t_{n-1})}\mathbb{E}[\phi_n[S_T - k_n]^+ \mid S_{t_{n-1}} = s]$$

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- Rational buyer only pays instalment rate if $V_{Std} \ge k_{n-1}$
  shortly before instalment date option is worth $\max(V_{Std} - k_{n-1}, 0)$

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- Rational buyer only pays instalment rate if $V_{Std} \ge k_{n-1}$
  shortly before instalment date option is worth $\max(V_{Std} - k_{n-1}, 0)$
- Compound option price at time t_{n-2} is

$$V_{n-1}(s) \overset{def}{=} V_{Cp}(s) = e^{-r_d(t_{n-1} - t_{n-2})}\mathbb{E}[\phi_{n-1}[V_n - k_{n-1}]^+ \mid S_{t_{n-2}} = s]$$

- Next steps are analogous, compound option V_i with option V_{i+1} so that V_i is an option on V_{i+1} with strike k_i and decision date t_i

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- Exact expression for value function of Instalment Option

$$V_i(s) \overset{def}{=} e^{-r_d(t_i - t_{i-1})}\mathbb{E}[(\phi_i(V_{i+1}(t_{i+1}) - k_i))^+ \mid S_{i-1} = s], \text{ for } i = 1, ..., n-1.$$

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- When carried out for all $i \leq n-1$, result is first instalment which is paid to open the deal at $t_0 = 0$

  $$p \overset{def}{=} V_1(s) = e^{-r_d(t_1-t_0)}\mathbb{E}[(\phi_1[V_2 - k_1])^+ \mid S_{t_0} = s]$$

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- Nested expectations require analysis of multiple integrals
- Numerical computation of multiple integrals is time consuming and possibly imprecise

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2.1. n-variate Cumulative Normal Formula

- n-variate cumulative normal function

$$\begin{aligned} N_n(h_i; \{\rho_{ij}\}_{1\leq j\leq n, i<j}) &= \text{Prob}\{Z_i < h_i; i = 1, ..., n\} \\ &= \int_{-\infty}^{h_1} ... \int_{-\infty}^{h_n} n(x_1, ..., x_n) dx_n ... dx_1 \end{aligned}$$

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- Curnow and Dunett (1962), see [6], show

$$N_n(h_i;\{\rho_{ij}\}) = \int_{-\infty}^{h_1} N_{n-1}\left(\frac{h_i-\rho_{i1}y}{(1-\delta_{i1}^2)^{\frac{1}{2}}};\{\rho_{ij*1}\}\right)n(y)dy \qquad i=2,...,n$$

$\rho_{ij*1} = \frac{\rho_{ij}-\rho_{i1}\rho_{j1}}{(1-\delta_{i1}^2)^{\frac{1}{2}}(1-\delta_{j1}^2)^{\frac{1}{2}}}$ $(i,j\neq 1 \text{ and } j\neq i)$

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- Special case $n = 2$ was used for compound option formula

$$N_2(h_1,h_2;\rho) = \int_{-\infty}^{h_1} N\left(\frac{h_2-\rho y}{(1-\rho^2)^{\frac{1}{2}}}\right) n(y)dy$$

$$
\begin{aligned}
V_{Cp} &= e^{r_f t_2} S_0 N_2 \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(+)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(+)} t_2}{\sigma \sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}} \right] \\
&- e^{-r_d t_2} k_2 N_2 \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(-)} t_2}{\sigma \sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}} \right] \\
&- e^{-r_d t_1} k_1 N \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma \sqrt{t_1}} \right]
\end{aligned}
$$

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$$\begin{aligned} V_{Cp} &= e^{r_f t_2} S_0 N_2 \left[\frac{\ln \frac{S_0}{S_1} + \mu^{(+)} t_1}{\sigma\sqrt{t_1}}, \frac{\ln \frac{S_0}{S_2} + \mu^{(+)} t_2}{\sigma\sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}} \right] \\ &- e^{-r_d t_2} k_2 N_2 \left[\frac{\ln \frac{S_0}{S_1} + \mu^{(-)} t_1}{\sigma\sqrt{t_1}}, \frac{\ln \frac{S_0}{S_2} + \mu^{(-)} t_2}{\sigma\sqrt{t_2}}, \sqrt{\frac{t_1}{t_2}} \right] \\ &- e^{-r_d t_1} k_1 N \left[\frac{\ln \frac{S_0}{S_1} + \mu^{(-)} t_1}{\sigma\sqrt{t_1}} \right] \end{aligned}$$

n-variate case

- $\vec{k} = (k_1, ..., k_n)$ strike prices
- $\vec{t} = (t_1, ..., t_n)$ instalment dates
- $\vec{\phi} = (\phi_1, ..., \phi_n)$ put/call indicators
- correlation coefficients of n-variate cumulative normal functions

$$\rho_{ij} = \sqrt{t_i/t_j} \text{ for } i, j = 1, ..., n \text{ and } i < j$$

$$
\begin{aligned}
& V_n(S_0, \vec{k}, \vec{t}, \sigma, r_d, r_f, \vec{\phi}) \\
= \quad & e^{-r_f t_n} S_0 \phi_1 \cdot \ldots \cdot \phi_n \\
& N_n \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(+)} t_1}{\sigma\sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(+)} t_2}{\sigma\sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{\bar{S}_n} + \mu^{(+)} t_n}{\sigma\sqrt{t_n}}; \{\rho_{ij}\} \right] \\
- \quad & e^{-r_d t_n} k_n \phi_1 \cdot \ldots \cdot \phi_n \\
& N_n \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma\sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(-)} t_2}{\sigma\sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{\bar{S}_n} + \mu^{(-)} t_n}{\sigma\sqrt{t_n}}; \{\rho_{ij}\} \right] \\
- \quad & e^{-r_d t_{n-1}} k_{n-1} \phi_1 \cdot \ldots \cdot \phi_{n-1} \\
& N_{n-1} \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma\sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(-)} t_2}{\sigma\sqrt{t_2}}, \ldots, \frac{\ln \frac{S_0}{\bar{S}_{n-1}} + \mu^{(-)} t_{n-1}}{\sigma\sqrt{t_{n-1}}}; \{\rho_{ij}\} \right] \\
& \vdots \\
- \quad & e^{-r_d t_2} k_2 \phi_1 \phi_2 N_2 \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma\sqrt{t_1}}, \frac{\ln \frac{S_0}{\bar{S}_2} + \mu^{(-)} t_2}{\sigma\sqrt{t_2}}; \rho_{12} \right] \\
- \quad & e^{-r_d t_1} k_1 \phi_1 N \left[\frac{\ln \frac{S_0}{\bar{S}_1} + \mu^{(-)} t_1}{\sigma\sqrt{t_1}} \right]
\end{aligned}
$$

A similar result has been independently derived by Thomassen and Wouve [21].

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2.2. Binomial Tree Option Pricing Technique

- Original work by Cox, Ross and Rubinstein (CRR) [5]
- Price movements of log-returns of underlying are modeled as constant up and down movements ($u = \exp(\sigma\sqrt{T/m}$, $d = \exp(-\sigma\sqrt{T/m})$ in the tree.
- Plain CRR suffers from oscillations (See Figure 6).
- Improvement: Leisen-Reimer Trees, see [14]:

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- Plain CRR suffers from oscillations (See Figure 6).
- Improvement: Leisen-Reimer Trees, see [14]:
- Replace p (probability of up move) by $p(d_-)$ using Peizer-Pratt Formula for odd n

$$p(z) = \frac{1}{2} + \operatorname{sign}(z)\frac{1}{2}\sqrt{1 - \exp\left[-\left(\frac{z}{n + \frac{1}{3} + \frac{1}{10\cdot(n+1)}}\right)^2\left(n + \frac{1}{6}\right)\right]},$$

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$$p(z) = \frac{1}{2} + \operatorname{sign}(z)\frac{1}{2}\sqrt{1 - \exp\left[-\left(\frac{z}{n + \frac{1}{3} + \frac{1}{10\cdot(n+1)}}\right)^2 \left(n + \frac{1}{6}\right)\right]},$$

and then

$$\begin{aligned} u &= e^{(r_d - r_f)\Delta t}\frac{p(d_+)}{p(d_-)} \\ d &= \frac{e^{(r_d - r_f)\Delta t} - p(d_-)u}{1 - p(d_-)} \\ d_\pm &= \frac{\ln\frac{S_0}{K} + (r_d - r_f \pm \frac{1}{2}\sigma)T}{\sigma\sqrt{T}} \end{aligned}$$

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2.3. Algorithm for Pricing Instalment Options by H. Ben-Ameur, M. Breton and P. François [2]

- Approximation of value of Instalment Option at t_0 through piecewise linear interpolation, therefore solving dynamic programming equation which results in a closed form

- Exercise value is $V_n(s) = \max(0, \phi_n(S_T - k_n))$

- Holding value at t_i is $V_i^h(s) = \mathbb{E}[e^{-r_d \Delta t} V_{i+1}(S_{t_{i+1}}) \mid S_{t_i} = s]$ for $i = 0, ..., n-1$
  where

$$V_i(s) = \begin{cases} V_0^h(s) & \text{for } i = 0 \\ \max(0, V_i^h(s) - k_i) & \text{for } i = 1, ..., n-1 \\ V_n(s) & \text{for } i = n \end{cases}$$

- Net holding value $V_i^h(s) - k_i$

- $a_0 = 0 < a_1 < \ldots < a_p < a_{p+1} = +\infty$ set of points $R_0, \ldots, R_p$ partition of $\mathbb{R}^+$ in $(p+1)$ intervals $R_j = (a_j, a_{j+1}]$ for $j = 0, \ldots, p$
- Given approximations $\tilde{V}_i$ of option value V_i at a_j in step i piecewise linear interpolation of this function achieved through

$$\hat{V}_i(s) = \sum_{i=0}^{p} (\alpha_j^i + \beta_j^i s) I_{a_j < s \le a_{j+1}}, \quad \tilde{V}_i(a_j) = \hat{V}_i(a_j), \quad \text{for } j = 0, \ldots, p-1,$$

$$\text{for } j = p \text{ choose } \alpha_p^i = \alpha_{p-1}^i \text{ and } \beta_p^i = \beta_{p-1}^i$$

- Assuming $\hat{V}_{i+1}$ is known, calculate expectation in step i

$$\begin{aligned}
\tilde{V}_i^h(a_k) &= \mathbb{E}[e^{-r_d \Delta t} \hat{V}_{i+1}(S_{t_{i+1}}) | S_{t_i} = a_k] \\
&= e^{-r_d \Delta t} \sum_{j=0}^{p} \alpha_j^{i+1} \mathbb{E}[I_{\frac{a_j}{a_k} < e^{\mu \Delta t + \sigma \sqrt{\Delta t} z} \le \frac{a_{j+1}}{a_k}}] \\
&+ \beta_j^{i+1} a_k \mathbb{E}[e^{\mu \Delta t + \sigma \sqrt{\Delta t} z} I_{\frac{a_j}{a_k} < e^{\mu \Delta t + \sigma \sqrt{\Delta t} z} \le \frac{a_{j+1}}{a_k}}],
\end{aligned}$$

$\mu = r_d - r_f - \sigma^2/2$, $\tilde{V}_i$ approximated holding value of Instalment Option

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- For $k = 1, ..., p$ and $j = 0, ..., p$ first integrals

$$A_{k,j} = \mathbb{E}[I_{\frac{a_j}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t} z} \le \frac{a_{j+1}}{a_k}}] \quad = \quad \begin{cases} N(x_{k,1}) & \text{for } j = 0 \\ N(x_{k,j+1}) - N(x_{k,j}) & \text{for } 1 \le j \le p-1 \\ 1 - N(x_{k,p}) & \text{for } j = p \end{cases}$$

$$\begin{aligned} B_{k,j} &= \mathbb{E}[a_k e^{\mu\Delta t + \sigma\sqrt{\Delta t} z} I_{\frac{a_j}{a_k} < e^{\mu\Delta t + \sigma\sqrt{\Delta t} z} \le \frac{a_{j+1}}{a_k}}] \\ &= \begin{cases} a_k N(x_{k,1} - \sigma\sqrt{\Delta t}) e^{(r_d - r_f)\Delta t} & \text{for} \quad j = 0 \\ a_k [N(x_{k,j+1} - \sigma\sqrt{\Delta t}) - N(x_{k,j} - \sigma\sqrt{\Delta t})] e^{(r_d - r_f)\Delta t} & \text{for} \quad 1 \le j \le p-1 \\ a_k [1 - N(x_{k,p} - \sigma\sqrt{\Delta t})] e^{(r_d - r_f)\Delta t} & \text{for} \quad j = p \end{cases} \end{aligned}$$

with $x_{k,j} = [\ln(a_j/a_k) - \mu\Delta t]/(\sigma\sqrt{\Delta t})$.

Procedure

0. Calculate a_i
1. Calculate $\hat{V}_n(s)$ for all s
2. Calculate $\tilde{V}^h_{n-1}(a_k)$ for all k in closed form
3. Calculate $\tilde{V}_{n-1}(a_k)$ for all k
4. Calculate $\hat{V}_{n-1}(s)$ for all $s > 0$
5. Iterate these steps until $\hat{V}_1(s_0)$=Price of Instalment Option at time 0 is calculated

3. Instalment Options with a Continuous Payment Plan

- Now examine what happens if we make the difference between the instalment payment dates t_i smaller.

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- In the limiting case the holder of the continuous instalment plan keeps paying at a rate p per time unit until she decides to terminate the contract.

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- In the limiting case it appears also intuitively obvious that the instalment plan is equivalent to the corresponding Vanilla plus a right to return it any time at a pre-specified rate, which is equivalent to the somehow discounted cumulative prolongation payment which one would have to pay for the time after termination.

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- We will now formalize this intuitive idea.

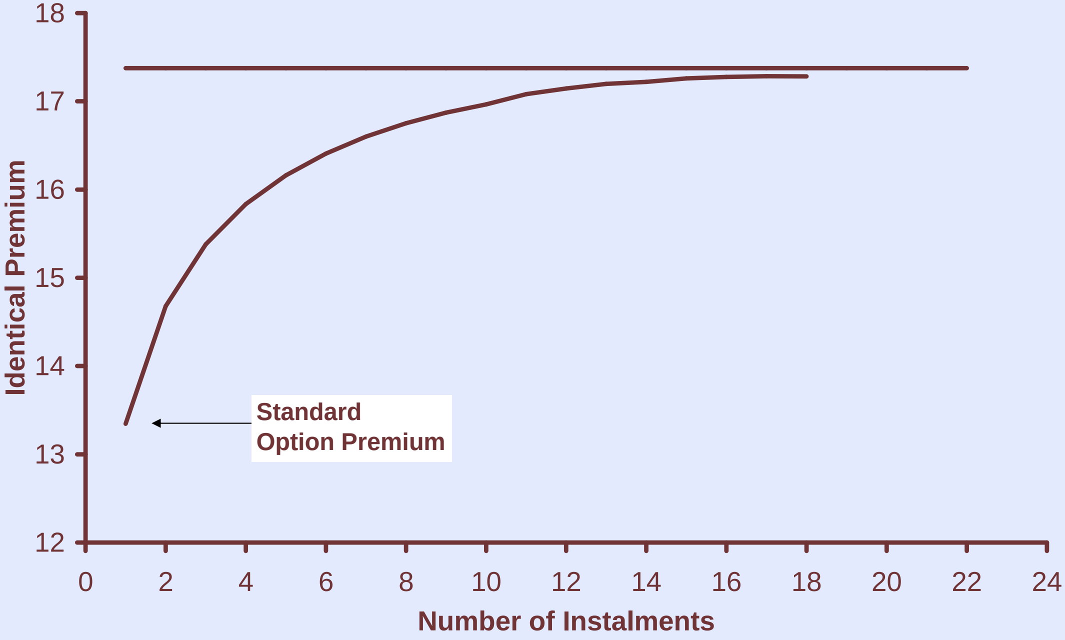

Figure 4: Convergence of uniform premium in discrete case to continuous premium. We have used the data $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r_d = 0.05$, $r_f = 0$, $T = 1$.

- Let $g = (g_t)_{t \in [0,T]}$ be the stochastic process describing the discounted net payoff of an Instalment option expressed as multiples of the domestic currency. If the holder stops paying the premium at time t, the difference between the option payoff and premium payments (all discounted to time 0) amounts to

$$g(t) \quad = \quad \begin{cases} e^{-r_d T}(S_T - K)^+ \mathbf{1}_{(t=T)} - \frac{p}{r_d}(1 - e^{-r_d t}) & \text{if } r_d \neq 0 \\ (S_T - K)^+ \mathbf{1}_{(t=T)} - pt & \text{if } r_d = 0 \end{cases} \tag{1}$$

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  where K is the strike.

- Given the premium rate p, there is a unique no-arbitrage premium P_0 to be paid at time 0 (supplementary to the rate p) given by

$$P_0 = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q(g_\tau), \qquad (2)$$

  where Q denotes the risk-neutral measure. Ideally, p is chosen as the *minimal* rate such that

$$P_0 = 0. \qquad (3)$$

- Note that P_0 from Equation (2) can never become negative as it is always possible to stop payments immediately. Thus, besides (3), we need a minimality assumption to obtain a unique rate.

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- Note that P_0 from Equation (2) can never become negative as it is always possible to stop payments immediately. Thus, besides (3), we need a minimality assumption to obtain a unique rate.

- We want to compare the Instalment option with the American contingent claim $f = (f_t)_{t\in[0,T]}$ given by

$$f_t = e^{-r_d t}(K_t - C_E(T-t, S_t))^+, \quad t \in [0, T], \tag{4}$$

  where $K_t = \frac{p}{r_d}\left(1 - e^{-r_d(T-t)}\right)$ for $r_d \neq 0$ and $K_t = p(T-t)$ when $r_d = 0$. C_E is the value of a standard European Call.

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- Equation (4) represents the payoff of an American Put on a European Call where the variable strike K_t of the Put equals the part of the instalments *not* to be paid if the holder decides to terminate the contract at time t. Define by $\tilde{f} = (\tilde{f}_t)_{t\in[0,T]}$ a similar American contingent claim with
$$\tilde{f}(t) = e^{-r_d t}\left[(K_t - C_E(T-t, S_t))^+ + C_E(T-t, S_t)\right], \quad t \in [0,T]. \tag{5}$$

- As the process $t \mapsto e^{-r_d t} C_E(T-t, S_t)$ is a Q-martingale we obtain that
$$\sup_{\tau \in \mathcal{T}_{0,T}} I\!\!E_Q(\tilde{f}_\tau) = C_E(T, s_0) + \sup_{\tau \in \mathcal{T}_{0,T}} I\!\!E_Q(f_\tau). \tag{6}$$

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- As the process $t \mapsto e^{-r_d t} C_E(T-t, S_t)$ is a Q-martingale we obtain that
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- We prove the following theorem using earlier results of El Karoui, Lepeltier and Millet in [12].

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- We prove the following theorem using earlier results of El Karoui, Lepeltier and Millet in [12].

Theorem 3.1

An Instalment option is the sum of a European Call plus an American Put on this European Call, i.e.

$$\underbrace{P_0 + p \int_0^T e^{-r_d s}\, ds}_{\textit{total premium payments}} = C_E(T, s_0) + \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}_Q(f_\tau)$$

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Proof.

- Define a new claim $\widetilde{g} = (\widetilde{g}_t)_{t\in[0,T]}$ differing from g only by a constant, namely $\widetilde{g}(t) = g(t) + p\int_0^T e^{-r_d s}\,ds$.

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- In view of (6) we have to show that

$$\sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q[\widetilde{g}(\tau)] = \sup_{\tau\in\mathcal{T}_{0,T}} \mathbb{E}_Q[\widetilde{f}(\tau)]. \tag{7}$$

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- The inequality with $\leq$ in (7) is obvious as we have $\widetilde{g} \leq \widetilde{f}$ pointwise.

- Let us show the other direction. Denote by $V = (V_t)_{t\in[0,T]}$ the Snell envelope of the potentially larger process $\widetilde{f}$, i.e. V is a càdlàg process (right continuous paths with left limits) with

$$V_t = \text{ess.sup}_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}_Q[\widetilde{f}(\tau) \mid \mathcal{F}_t], \quad P\text{-a.s.}, \quad t \in [0,T],$$

where $(\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration of the process S.

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  where $(\mathcal{F}_t)_{t\in[0,T]}$ is the canonical filtration of the process S.

- Define by $h = h(u,s)$ the value of the Instalment option (with rate p) if the initial price of the underlying is $s \in \mathbb{R}_+$ and time to maturity of the contract is $u \in \mathbb{R}_+$, i.e.

$$h(u,s) \;=\; \sup_{\tau\in\mathcal{T}_{0,u}} \mathbb{E}_s \left[e^{-r_d\tau} \left[\left[\frac{p}{r_d}\left(1 - e^{-r_d(u-\tau)}\right) - C_E(u-\tau, \widetilde{S}_\tau) \right]^+ + C_E(u-\tau, \widetilde{S}_\tau) \right] \right]$$

where $\widetilde{S}$ is again a geometric Brownian motion with the same probabilistic characteristics as S.

- Using the Markov property of S we can apply Theorem 3.4 in El Karoui/Lepeltier/Millet (1992) and obtain

$$\begin{aligned} V_t &= \text{ess.sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_Q \left[\widetilde{f}(\tau) \mid \mathcal{F}_t \right] \\ &= \text{ess.sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_Q \left[\widetilde{f}(\tau) \mid S_t \right] \\ &= e^{-r_d t} h(T - t, S_t). \end{aligned}$$

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- As $\widetilde{f}$ has continuous paths the optimal exercise time after t is given by

$$\begin{aligned} \tau^t &= \inf\{u \in [t,T] \mid V_u = \widetilde{f}(u)\} \\ &= \inf\{t \in [0,T] \mid e^{-r_d t} h(T-t, S_t) = \widetilde{f}(t)\}. \end{aligned} \tag{8}$$

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- Keeping this in mind, we want to show that

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- As the process $t \mapsto e^{-r_d t} C_E(T-t, S_t)$ is a martingale we can pull it out of the optimal stopping problem and obtain

$$h(u,s) = C_E(u,s) + \sup_{\tau \in \mathcal{T}_{0,u}} \mathbb{E}_s \left[e^{-r_d \tau} \left[\frac{p}{r_d}\left(1 - e^{-r_d(u-\tau)}\right) - C_E(u-\tau, \widetilde{S}_\tau) \right]^+ \right],$$

  and thus $h(u,s) > C_E(u,s)$, for all $u>0, s>0$, as the underlying $C_E(u, \widetilde{S}_t)$ can always get into the money with positive probability as long as $u>0$.

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- Therefore, we obtain for $t \in [0,T)$ and $s \in (0,\infty)$ the following implication

$$h(T-t,s) = (K_t - C_E(T-t,s))^+ + C_E(T-t,s) \Rightarrow K_t > C_E(T-t,s). \tag{10}$$

- This means that by (8) $\widetilde{f}$ is only exercised prematurely when $K_t > C_E(T-t, S_t)$.

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- As at maturity the payoffs of $\widetilde{f}$ and $\widetilde{g}$ coincide anyway, we have for the optimal exercise time τ^0 of the process $\widetilde{f}$

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- Therefore we arrive at (7) and the assertion of the theorem follows.

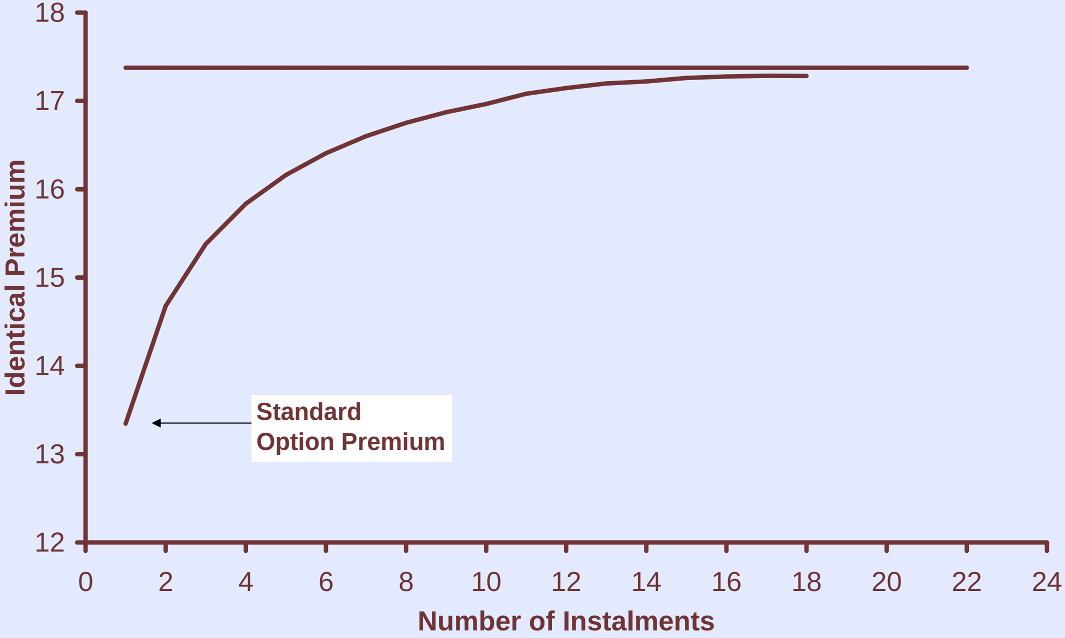

Figure 5: Convergence of uniform premium in discrete case to continuous premium. We have used the data $S_0 = 100$, $K = 95$, $\sigma = 0.2$, $r_d = 0.05$, $r_f = 0$, $T = 1$.

4. Numerical Results

- Results of binomial trees oscillate strongly

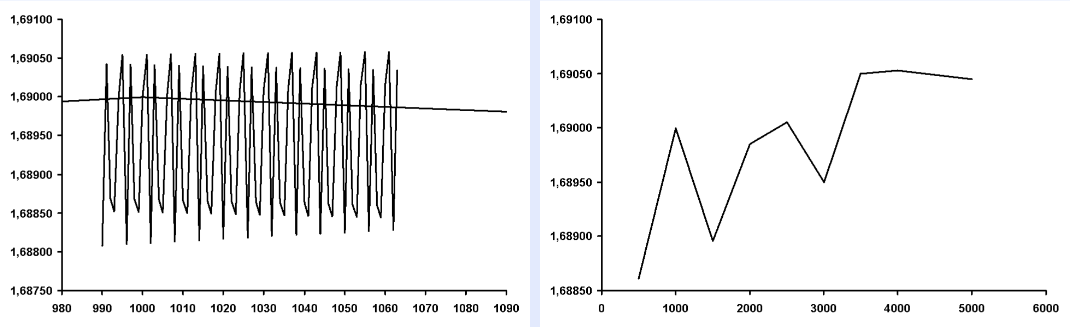

Figure 6: Convergence of the value function in the binomial trees implementation

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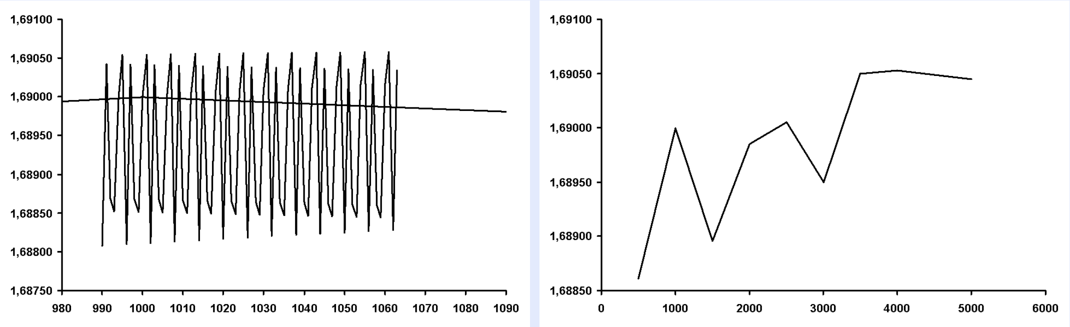

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- n-variate formula is the fastest of all considered methods, even for higher numbers of instalments.

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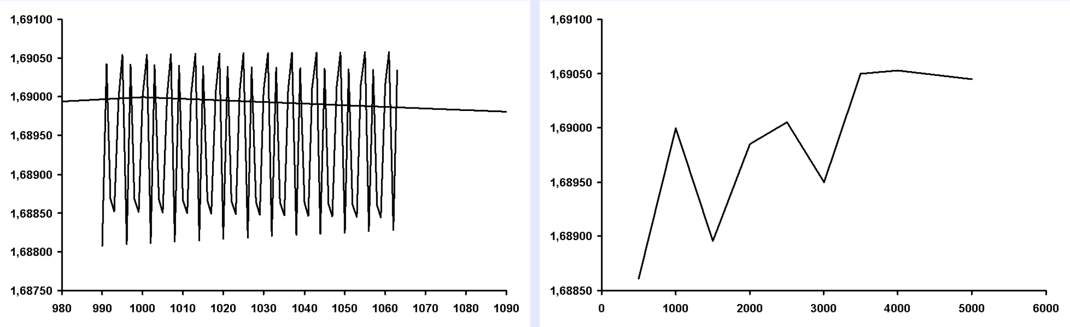

Figure 6: Convergence of the value function in the binomial trees implementation

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- Accuracy of n-variate formula now only depends on accuracy of calculation of multivariate normal integrals and calculation of roots.

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4. Numerical Results

- Results of binomial trees oscillate strongly

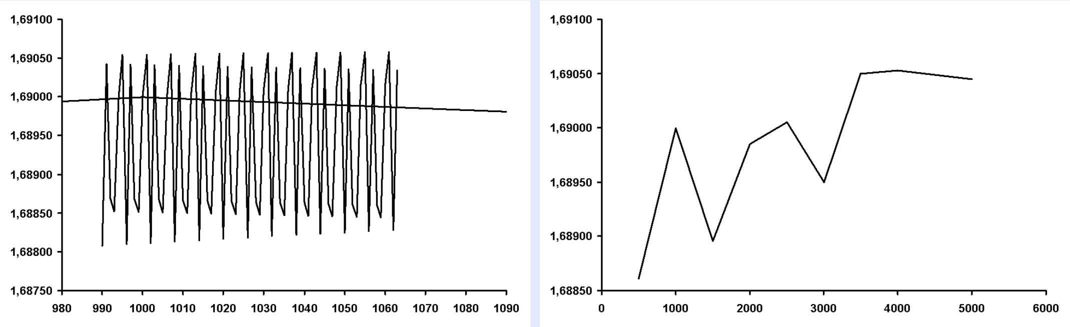

Figure 6: Convergence of the value function in the binomial trees implementation

- n-variate formula is the fastest of all considered methods, even for higher numbers of instalments.
- Accuracy of n-variate formula now only depends on accuracy of calculation of multivariate normal integrals and calculation of roots.
- Algorithm of BBF works for equally distant instalment dates

Performance

Numerical Method	**Value of V_3**		**CPU Time**
Binomial trees for $n = 4000$	1.69053	0.0137335	1109
Closed-form formula for $n = 3$	1.69092	0.0137339	< 1
BBF-Algorithm [2] with $p = 4000$	1.69084	0.0137332	168
50000-point Gauß-Legendre integration	1.69087	0.0137339	176
Numerical integration with Mathematica	1.69091	0.0137299	47

Table 1: Performance comparison of Instalment valuation algorithms.
Scenario 1: $S_0 = 100$, $k_1 = 100$, $k_{2,3} = 3$, $\sigma = 20\%$, $r_d = 10\%$, $r_f = 15\%$, $T = 1$, $\Delta t = 1/3$, $\phi_{1,2,3} = 1$.
Scenario 2: $S_0 = 1.15$, $k_1 = 1.15$, $k_{2,3} = 0.02$, $\sigma = 10\%$, $r_d = 1\%$, $r_f = 2\%$, $T = 1,t = 1/3$, $\phi_{1,2,3} = 1$.

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4.1. The Mathematica Package `instalment.m`

```
BeginPackage["Options`Instalment`"]

Instalment::usage = "Instalment[S,K,T,vol,rd,rf,phi,N] \n
Black-Scholes value for European Instalment options\n
S: spot\n
K: strike list of individual options\n
T: time differences to maturity in years between individual options\n
beginning with Vanilla option maturity\n
vol:  volatility\n
rd:  domestic risk free rate:  discounting is done as Exp[-T[[i]]*rd] \n
rf:  foreign risk free rate:  discounting is done as Exp[-T[[i]]*rf]\n
phi:  list of +1 for Calls, -1 for Puts\n
N: number of options in Instalment option"

Begin["`Private`"]

ncum[x_] := 1/2*(Erf[x/Sqrt[2]] + 1); (*cumulative standard normal*)
ndf[x_] := Evaluate[D[ncum[x], x]]; (*standard normal density*)

Vanilla[x_, K_, vol_, r_, rf_, T_, fi_]:=Block[dp, dm,
    dp = (Log[x/K] + (r- rf + 0.5*vol*vol)*T)/(vol*Sqrt[T]);
    dm = (Log[x/K] + (r - rf - 0.5*vol*vol)*T)/(vol*Sqrt[T]);
    fi*(Exp[-rf*T]*x*ncum[fi*dp] - Exp[-r*T]*K*ncum[fi*dm])];
```

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```
Instalment[S_, K_, T_, vol_, rd_, rf_, phi_, N_] := Block[mu,
          mu = rd - rf - 0.5*vol*vol;
    If[N == 1, Vanilla[S, K[[1]], vol, rd, rf, T[[1]], phi[[1]]],
          Exp[-T[[N]]*rd]*
    NIntegrate[
          Max[0, phi[[N]]*(Instalment[S*Exp[vol*Sqrt[T[[N]]]*z + mu* T[[N]]
             K, T, vol, rd, rf, phi, N - 1] -
          K[[N]])]*ndf[z], z, -10, 10]]];

End[]
EndPackage[]
```

4.1.1. The Testing Environment `instalment_testenv.nb`

```
spot = 100
vol = 0.2
tau = {1/3, 1/3, 1/3}
rd = 0.10
rf = 0.15
strike = {100, 3, 3}
phi = {1, 1, 1}

Instalment[spot, strike, tau, vol, rd, rf, phi, 3]

1.69085
```

5. Contact Information

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